

ZIEGLER SPECTRUM AND KRULL GABRIEL DIMENSION

DANIEL LÓPEZ-AGUAYO

ABSTRACT. These notes are based on a talk given at the Summer School “*Infinite-dimensional representations of finite-dimensional algebras*” held at the University of Manchester in September 2015. They intend to provide a brief introduction to the notion of Ziegler Spectrum and Krull-Gabriel dimension.

Let k be a field and let A be an associative k -algebra. Let $\text{Mod-}A$ (respectively $A\text{-Mod}$) denote the category of right A -modules (respectively left A -modules) and let $D: A\text{-Mod} \rightarrow \text{Mod-}A$ be the functor $D(-) = \text{Hom}_k(-, k)$. Throughout these notes \mathbf{Ab} will denote the category of abelian groups.

Definition 1. A monomorphism $M \xrightarrow{f} N$ in $\text{Mod-}A$ is pure if after tensoring with every left A -module remains injective. More precisely, if $f \otimes 1_X : M \otimes_A X \rightarrow N \otimes_A X$ is injective for each left A -module X .

Definition 2. A module M is pure injective if every pure monomorphism $f : M \hookrightarrow L$ splits, i.e. there exists an A -module map $g : L \rightarrow M$ such that $gf = id_M$.

Definition 3. Let $\mathcal{E} : 0 \rightarrow B \xrightarrow{\alpha} C \xrightarrow{\beta} D \rightarrow 0$ be a short exact sequence in $\text{Mod-}A$. We say \mathcal{E} is pure-exact if for every left A -module X the homomorphism $B \otimes_A X \xrightarrow{\alpha \otimes 1_X} C \otimes_A X$ is injective.

Remark 1. If the contravariant functor $\text{Hom}_A(-, Q)$ is exact on pure exact sequences starting at Q then Q is pure injective.

Example 1. Duals are always pure-injective: let M be a left A -module and let $0 \rightarrow D(M) \rightarrow N \rightarrow L \rightarrow 0$ be pure-exact. Then $0 \rightarrow D(M) \otimes_A M \rightarrow N \otimes_A M \rightarrow L \otimes_A M \rightarrow 0$ is exact. Applying the contravariant exact functor $\text{Hom}_k(-, k)$ yields the following short exact sequence:

$$0 \rightarrow \text{Hom}_k(L \otimes_A M, k) \rightarrow \text{Hom}_k(N \otimes_A M, k) \rightarrow \text{Hom}_k(D(M) \otimes_A M, k) \rightarrow 0$$

Recall that $\text{Hom}_A(-, D(M)) \cong \text{Hom}_k(- \otimes_A M, k)$. Thus we get a short exact sequence:

$$0 \rightarrow \text{Hom}_A(L, D(M)) \rightarrow \text{Hom}_A(N, D(M)) \rightarrow \text{Hom}_A(D(M), D(M)) \rightarrow 0$$

It follows that $D(M)$ is pure exact. In particular, any finite dimensional module is pure injective (as $M \cong D^2(M)$).

Definition 4. An indecomposable A -module of infinite dimension is said to be **generic** provided M considered as an $\text{End}_A(M)$ -module is of finite length (i.e. endofinite).

Example 2. An example of a generic module is the following representation of the Kronecker quiver:

$$\begin{array}{ccc} & \xrightarrow{id} & \\ k(T) & & k(T) \\ & \xleftarrow{T} & \end{array}$$

here, $k(T)$ denotes the field of rational functions in one variable T over the base field k .

Example 3. Any endofinite module is pure-injective. In particular, generic modules are pure-injective.

Example 4. The evaluation map $M \rightarrow M^{**}$ is a pure monomorphism. See [9, Corollary 1.3.16].

Definition 5. The Ziegler spectrum of an algebra A , ZgA , is defined to be the set of isoclasses of indecomposable pure-injective A -modules. Later, we will see that this set can be endowed with a topology.

Example 5. Up to isomorphism, the indecomposable pure injective \mathbb{Z} -modules [5] are:

- (a) finite groups $\mathbb{Z}/p^n\mathbb{Z}$,
- (b) Prüfer groups \mathbb{Z}_{p^∞} ,
- (c) the p -adic integers $\overline{\mathbb{Z}_{(p)}}$,
- (d) \mathbb{Q}

where p ranges over primes and n over positive integers. This determines completely $Zg\mathbb{Z}$. Remark that the fact that (c) and (d) are pure injective follows from the well known facts that these are divisible abelian groups (recall that a \mathbb{Z} -module is injective iff it is divisible) thus they are injective and therefore pure injective.

Remark 2. Some facts about the Ziegler spectrum of a finite dimensional algebra (see [6]):

- (a) $\{M\}$ is open if and only if M is finite dimensional. This can be shown using Auslander-Reiten theory.
- (b) The finite dimensional points form a dense subset.
- (c) If R is of finite representation type then the ZgR is discrete (i.e. has the discrete topology). [If R is of finite representation type then by [9, Proposition 4.5.22] it follows that every indecomposable module is endofinite and thus by [9, Proposition 5.1.12] such a module is a closed point of ZgR . Since there are finitely many points in ZgR , the claim follows.]

Let \mathcal{A} be an additive category. Given an object $X \in \mathcal{A}$ we denote by H_X the representable functor:

$$H_X := \text{Hom}_{\mathcal{A}}(X, -) : \mathcal{A} \rightarrow \mathbf{Ab}, Y \mapsto \text{Hom}_{\mathcal{A}}(X, Y).$$

Definition 6. A functor $F : \mathcal{A} \rightarrow \mathbf{Ab}$ is finitely presented if there exists an exact sequence:

$$H_Y \rightarrow H_X \rightarrow F \rightarrow 0$$

in the functor category $(\mathcal{A}, \mathbf{Ab})$.

Definition 7. An A -module M is finitely presented if there is an exact sequence:

$$A^m \rightarrow A^n \rightarrow M \rightarrow 0$$

where $m, n \in \mathbb{N}$.

Theorem 1. [1, 6.1] *Let M be a right R -module. The functor $M \otimes_R - : R\text{-mod} \rightarrow \mathbf{Ab}$ is finitely presented if and only if M is finitely presented.*

Definition 8. An additive functor $F: \text{Mod-}A \rightarrow \mathbf{Ab}$ is said to be coherent if it commutes with direct limits and products. The coherent functors form an abelian category whose morphisms are the natural transformations.

In the next list, modules are taken in the category $\text{Mod-}A$ where A is a finite dimensional k - algebra. The following list provides examples of coherent functors [3].

- (1) The representable functor $\text{Hom}_A(M, -)$ where M is finite dimensional.
- (2) The tensor product $M \otimes_A -$ where M is a finite-dimensional right A -module.
- (3) The derived functor $\text{Ext}^i(X, -)$ where X is finite dimensional.
- (4) The functor $\text{Tor}_i(N, -)$ where N is finite dimensional.

We now state a theorem which gives a way of putting a topology on ZgA via coherent functors. We won't assume compact spaces are Hausdorff; in fact the Ziegler spectrum is seldom Hausdorff [9, 8.2.12]

Theorem 2. (Ziegler) *The sets $\{M \in ZgA : F(M) \neq 0\}$ with F a coherent functor, form a base of open sets for a topology on ZgA . With this topology ZgA becomes a compact topological space.*

Serre subcategories and localization

Definition 9. Let \mathcal{A} be an abelian category. Let \mathcal{S} be a full subcategory of \mathcal{A} . We say that \mathcal{S} is a Serre subcategory (or a Serre class) of \mathcal{A} such that whenever $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence in \mathcal{A} then $B \in \mathcal{S}$ if and only if $A, C \in \mathcal{S}$.

Example 6. All torsion groups, all finitely generated groups, all finite groups, all p -groups in the category \mathbf{Ab} are Serre subcategories.

Example 7. The subclass of $\text{Mod-}A$ consisting of Noetherian (resp. Artinian) right A -modules is a Serre class.

Example 8. If $F: \mathcal{A} \rightarrow \mathcal{B}$ is a functor, then the kernel of F is defined to be the full subcategory with objects:

$$\text{Ker}(F) = \{A \in \text{ob}(\mathcal{A}) : F(A) = 0\}$$

If F is exact, then the kernel of F is a Serre class. This is immediate from the fact that if a sequence $0 \rightarrow F(X) \rightarrow 0$ is exact if and only if $F(X) = 0$ i.e iff $X \in \text{ker}(F)$.

Example 9. Check that the class $\{M \in R\text{-Mod} : I^n M = 0 \text{ for some } n \geq 1\}$, for a fixed ideal $I \subseteq R$, is Serre.

Let \mathcal{A} be an abelian category.

Definition 10. A subobject of an object X is an object Y together with a monomorphism $i : Y \hookrightarrow X$.

Definition 11. A quotient object of Y is an object Z with an epimorphism $p : Y \rightarrow Z$.

Definition 12. A subquotient object of Y is a quotient object of a subobject of Y .

Definition 13. For a subobject $X \subset Y$ define the quotient object $Z := Y/X$ to be the cokernel of a monomorphism $f : X \rightarrow Y$. We write $Y \subseteq X$.

The notion of a Serre class \mathcal{S} of \mathcal{A} will be used to define a **quotient category** in the sense of Serre-Grothendieck. We sketch the construction. See [4, pp.498-505] for more details.

Let A, B be objects of \mathcal{A} and let A', B' be subobjects of A and B respectively (i.e. $A' \subseteq A, B' \subseteq B$). Let I denote the class of ordered pairs $\langle A', B' \rangle$ where $A' \subseteq A, B' \subseteq B$ and $A/A', B' \in \mathcal{S}$. Order I by:

$$\langle A', B' \rangle \leq \langle A'', B'' \rangle \text{ iff } A' \supseteq A'' \text{ and } B' \subseteq B''.$$

One can verify that I is a directed set.

Note that if $\langle A', B' \rangle \leq \langle A'', B'' \rangle$ then there is an induced map:

$$\text{Hom}_{\mathcal{A}}(A', B/B') \rightarrow \text{Hom}_{\mathcal{A}}(A'', B/B') \rightarrow \text{Hom}_{\mathcal{A}}(A'', B/B'')$$

Then $\{\text{Hom}_{\mathcal{A}}(A', B/B') : A/A' \in \mathcal{S}, B' \in \mathcal{S}\}$ is a directed system of abelian groups indexed by I so it has a direct limit.

Let \mathcal{A} be an abelian category and let \mathcal{S} be a Serre subcategory. The **quotient category** \mathcal{A}/\mathcal{S} is defined as follows. The objects of \mathcal{A}/\mathcal{S} are the same objects of \mathcal{A} and the morphisms are defined as:

$$\text{Hom}_{\mathcal{A}/\mathcal{S}}(A, B) := \varinjlim (A', B/B')$$

where the limit is over subobjects $A' \subseteq A, B' \subseteq B$ such that $A/A', B' \in \mathcal{S}$.

There is clearly a **quotient functor** $\pi : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{S}$: we define $\pi(A) = A$ for every object $A \in \mathcal{A}$ and let $\pi(f : M \rightarrow N)$ be the image of f in the direct system that defines $\text{Hom}_{\mathcal{A}/\mathcal{S}}(M, N)$.

Example 10. The quotient of **Ab** modulo its Serre subcategory of torsion groups is the category of **Q**-vector spaces.

An alternative way of constructing the quotient category with respect a Serre class \mathcal{S} is to use localization where the multiplicative system is given by the set $C_{\mathcal{S}} = \{f \in \text{Mor}(\mathcal{A}) : \ker(f), \text{coker}(f) \in \text{Ob}(\mathcal{S})\}$. Then we let:

$$\mathcal{A}/\mathcal{S} := \mathcal{A}[C_{\mathcal{S}}^{-1}]$$

Definition 14. Let \mathcal{S} be a Serre subcategory. We say \mathcal{S} is a **localizing subcategory** of \mathcal{A} , if the quotient functor $T : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{S}$ has a right adjoint $G : \mathcal{A}/\mathcal{S} \rightarrow \mathcal{A}$; the functor G is called a section functor. This means that for every two objects $A \in \mathcal{A}, B \in \mathcal{A}/\mathcal{S}$ there are bijections:

$$\text{Hom}_{\mathcal{A}/\mathcal{S}}(T(A), B) \xrightleftharpoons[\phi_{A,B}]{\psi_{A,B}} \text{Hom}_{\mathcal{A}}(A, G(B))$$

natural in A and B . The functor T is also called the **localization functor**.

The following theorem ensures that \mathcal{A}/\mathcal{S} is an abelian category, so the quotient category is not the same as factoring by an ideal of \mathcal{S} (the process used to obtain the stable module category $\underline{\text{mod}} \Lambda$).

Theorem 3. *Let \mathcal{A} be an abelian category and let \mathcal{S} be a Serre subcategory of \mathcal{A} . The quotient category \mathcal{A}/\mathcal{S} is abelian and the localization functor $F : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{S}$ is exact. The kernel of $F = \{A \in \mathcal{A} : F(A) = 0\}$ is \mathcal{S} . Every exact functor $G : \mathcal{A} \rightarrow \mathcal{A}'$ where \mathcal{A}' is abelian category and $\mathcal{S} \subseteq \ker(G)$ factors uniquely through $F : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{S}$ via an exact and faithful functor.*

There is full and faithful embedding of $\text{Mod } -R$ into the functor category $(R - \text{mod}, \mathbf{Ab})$ where $R - \text{mod}$ is the category of finitely presented left R -modules which is given on objects by taking the R -module M_R to the tensor product $M \otimes_R - : R - \text{mod} \rightarrow \mathbf{Ab}$ (see [11, Theorem 2.2]).

Definition 15. Let \mathcal{C} be a preadditive category. A generating set for \mathcal{C} is a collection \mathcal{A} of objects of \mathcal{C} such that for every nonzero morphism $f : B \rightarrow C$ in \mathcal{C} there is $G \in \mathcal{A}$ and a morphism $g : G \rightarrow B$ such that $fg \neq 0$. If $\mathcal{A} = \{G\}$ then we say that G is a generator.

Example 11. In the category \mathbf{Ab} the abelian group \mathbb{Z} is a generator. In the category \mathbf{Set} any singleton is a generator.

Definition 16. A Grothendieck category is an abelian category which has arbitrary coproducts, in which direct limits are exact and which has a generator.

The following are examples of Grothendieck categories.

- (1) Given any associative unital ring R , the category $\text{Mod } -R$ is a Grothendieck category. In particular, \mathbf{Ab} is a Grothendieck category.
- (2) Given a small category \mathcal{C} and a Grothendieck category \mathcal{A} , the functor category $\text{Funct}(\mathcal{C}, \mathcal{A})$ is a Grothendieck category. If \mathcal{C} is preadditive, then the functor category $\text{Add}(\mathcal{C}, \mathcal{A})$ of all additive functors is a Grothendieck category as well.
- (3) If \mathcal{A} is a Grothendieck category and \mathcal{S} is a localizing subcategory of \mathcal{A} , then the Serre quotient category \mathcal{A}/\mathcal{S} is a Grothendieck category.

Definition 17. Let \mathcal{C} be an additive category with direct limits. An object A of a category \mathcal{C} is **finitely presented** if the representable functor $(A, -) := \text{Hom}_{\mathcal{C}}(A, -)$ commutes with direct limits, meaning that if $((C_{\lambda})_{\lambda}, (f_{\lambda\mu} : C_{\lambda} \rightarrow C_{\mu})_{\lambda < \mu})$ is any directed system in \mathcal{C} and if $(C, (f_{\lambda} : C_{\lambda} \rightarrow C)_{\lambda})$ is the direct limit of this system then the direct limit of the directed system $((A, C_{\lambda})_{\lambda}, (A, f_{\lambda\mu} : (A, C_{\lambda}) \rightarrow (A, C_{\mu}))_{\lambda < \mu})$ in \mathbf{Set} is $((A, C), (A, f_{\lambda})_{\lambda})$.

Let \mathcal{A} be a Grothendieck category. A full subcategory \mathcal{T} is a **torsion subcategory** if it is closed under extensions, quotient objects and arbitrary coproducts. If it is also closed under subobjects then it is a **hereditary** torsion subcategory (or subclass). The objects of \mathcal{T} are referred to as **torsion** and those in the corresponding torsionfree subcategory

$\mathcal{F} = \{D : (\mathcal{T}, D) = 0\} = \{D : \text{Hom}_{\mathcal{A}}(T, D) = 0 \text{ for all } T \text{ in } \mathcal{T}\}$ are the **torsionfree** objects. It is then the case that $\mathcal{T} = \{C : (C, \mathcal{F}) = 0\}$ and the pair $\tau = (\mathcal{T}, \mathcal{F})$ is referred to as a **torsion theory**, which is said to be **hereditary** if \mathcal{T} is closed under subobjects.

Definition 18. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a (hereditary) torsion theory on \mathcal{A} . We say τ is of **finite type** if the torsionfree class \mathcal{F} is closed under direct limits. Equivalently, if the torsion class \mathcal{T} is generated as such by the class of finitely presented torsion objects.

If \mathcal{A} is a functor category, then the closure under direct limits, $\overrightarrow{\mathcal{S}}$, of any Serre subcategory \mathcal{S} of \mathcal{A}^{fp} is a hereditary torsion class in \mathcal{A} , that is, a class closed under subobjects, factor objects, extensions and arbitrary coproducts.

Hereditary torsion theories, τ , of finite type on the functor category $(\text{mod} - \mathcal{A}, \mathbf{Ab})$ correspond bijectively to the Serre subcategories, \mathcal{S} , of the finitely presented functor category $\text{fun} - \mathcal{A} = (\text{mod} - \mathcal{A}, \mathbf{Ab})^{fp}$, the maps being $\tau \mapsto \mathcal{T}_\tau \cap \text{fun} - \mathcal{A}$ and the inverse map sends $\mathcal{S} \mapsto \overrightarrow{\mathcal{S}}$.

Definition 19. A Grothendieck category \mathcal{A} is said to be **locally coherent** provided that \mathcal{A} has a generating set of finitely presented objects and the full subcategory of finitely presented objects in \mathcal{A} is abelian.

Definition 20. Let X be an object in an abelian category \mathcal{C} . We say that X has finite length if there exists a filtration:

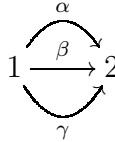
$$0 = X_0 \subset X_1 \subset \dots \subset X_{n-1} \subset X_n = X$$

such that X_i/X_{i-1} is simple for all i . Such a filtration is called a Jordan-Hölder series of X .

Let \mathcal{A} be a locally coherent Grothendieck category. Let \mathcal{S} be the Serre subcategory of all finitely presented objects of \mathcal{A} of finite length. Then $\overrightarrow{\mathcal{S}}$ is a finite type torsion class in \mathcal{A} and we may form the localization $\mathcal{A}_1 = \mathcal{A} / \overrightarrow{\mathcal{S}}$. By [11, Corollary 3.6] \mathcal{A}_1 is again a locally coherent category, so we can repeat the process with \mathcal{A}_1 in place of \mathcal{A} , and obtain \mathcal{A}_2 , etc, transfinitely. We obtain a sequence of locally coherent categories \mathcal{A}_α indexed by ordinals. If α is the smallest ordinal such that $\mathcal{A}_\alpha = 0$ then we say that the **Krull-Gabriel dimension** of \mathcal{A} is α and write $KGdim \mathcal{A} = \alpha$. If eventually we reach a category with no finitely presented simple objects then we set $KGdim(\mathcal{A}) = \infty$.

If \mathcal{A} is the functor category $(\text{mod} - R, \mathbf{Ab})$ then we write $KG(R)$ for $KGdim(\text{mod} - R, \mathbf{Ab})$. There is a connection between this dimension and representation type of Artin algebras [11, p.23]:

- (1) $KG(R) = 0$ if and only if R is of finite representation type. (e.g. the truncated polynomial algebra $R = k[x]/(x^n)$).
- (2) There is no Artin algebra whose Krull-Gabriel dimension is 1.
- (3) Algebras with Krull-Gabriel dimension 2 include the tame hereditary algebras (see [7]).
- (4) Wild-finite dimensional algebras have infinite Krull-Gabriel dimension (see [10, pp.281-2]); for example, the path algebra of the extended three Kronecker quiver:



has infinite Krull-Gabriel dimension.

- (5) For hereditary Artin algebras (an algebra of global dimension ≤ 1 that is finitely generated as a module over its Artinian center; e.g. the path algebra of an acyclic finite quiver) the possible representation types -finite,tame/domestic, wild- correspond to the values $0, 2, \infty$ for Krull-Gabriel dimension.
- (6) The domestic string algebras Λ_n have $KG(\Lambda_n) = n + 1$ (see [2]).

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CENTRO DE CIENCIAS MATEMÁTICAS UNAM, MORELIA, MEXICO
E-mail address: dlopez@matmor.unam.mx